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CONTRIBUTIONS TO THE HISTORY
OF THE AXIOM OF FOUNDATION

Abstract

The *axiom of foundation* is a principle asserting the non existence, in the domain of set theory, of what T. Skolem called “infinite descending $\in$ -sequences”, that is those infinite chains of sets of the form $\dots x_n \in x_{n-1} \in \dots \in x_1 \in x_0$, constituted from a set, an element of the set, an element of the element, and so on. In particular, the axiom of foundation excludes, in the universe of the theory, the existence of circular sets of the kind $x \in x$ and $x_0 \in x_1 \in \dots \in x_1 \in x_0$.

After the discovery of the Burali-Forti and of the Russell paradoxes, E. Zermelo solved the Russell paradox (in 1908) without denying the possibility of the existence of sets x such that $x \in x$, while B. Russell, with his *theory of types* (1908), rejects explicitly *circularity* and *non-well-foundedness* and proposes a universe structured in a hierarchy of levels, built up by *iterating* an infinite number of times the operations of set formation starting from a given collection of objects (iterative conception).

Between 1917 and 1920 D. Mirimanoff proposed an axiomatic system in which one can solve the Burali-Forti paradox and at the same time classify rigorously some distinct kind of non-well-founded sets.

However, from 1920 on Russell’s view became paradigmatic, and all except P. Finsler accepted it: the axiom of foundation, introduced by von Neumann in 1925 and then investigated in depth by Zermelo, is the formal counterpart of the intuition on which the theory of types and the iterative conception are based.

The aim of the present paper is to explain the emergence and the almost universal acceptance of the axiom of foundation in the early years of Zermelo-Skolem-Fraenkel axiomatic set theory.